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Best proximity point results for p-semi-cyclic contraction pair in Banach spaces

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Abstract. In the present work, we extend the semi-cyclic contraction conditions established by Gabeleh and Abkar [9] by introducing the concept of a p-semi-cyclic contraction for a pair of mappings (S, T) on the union of $p(\geq 2)$ nonempty subsets in a Banach space. In the sequel, we establish several fundamental results, including existence and convergence theorems for best proximity points corresponding to the pair (S, T). Additionally, illustrative examples are provided to support our findings.

1. Introduction

Solving equations of the type $Tx = x$, where T is a mapping defined on subsets of metric or normed linear spaces, heavily relies on fixed point theory. The Banach Contraction Principle, which was came to light in 1922 [3] and signaled the start of fixed point theory in metric spaces, is a basic conclusion in fixed point theory. Since then, fixed point theory has been widely studied by various authors (One may see [11, 12, 14, 15, 17]).

Furthermore, a fixed point is not always present in a non-self mapping $T : A \to B$. In such cases, the objective is often to find an element $x \in A$ such that, in some sense, approximate to its image $Tx \in B$. This leads to the development of best approximation and best proximity point theory, which serve as meaningful alternatives to classical fixed point results in the context of non-self mappings. A best approximation theorem deals with the existence of solutions that minimize the distance to the desired outcome, although such solutions may not always be globally optimal. In this direction, we consider the well-known best approximation theorem introduced by Ky Fan [8] in 1969, stated as follows:

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Theorem 1 ([8])**.** *Let P be a nonempty compact convex subset of a normed linear space X and $T : P \to X$ be a continuous function. Then, there exists $x \in A$ such that $\|x - Tx\| = d(Tx, A) = \inf\{\|Tx - a\| : a \in A\}$.*

In the above theorem, the element x does not necessarily attain the optimal value of $\|x - Tx\|$. In contrast, best proximity point theorems ensure optimal approximate solutions. More precisely, a best proximity point theorem ensures the existence of a point x such that $d(A, B) = d(x, Tx)$, where $d(A, B) = \min\{d(u, v) : u \in A, v \in B\}$. Over the last two decades, best proximity point theory has emerged as an active area of research. Some significant contributions in this direction can be found in references [3,7,18].

The concept of cyclic contraction mappings on uniformly convex Banach spaces was first presented in 2006 by Eldred and Veeremani [5], who also looked into the potential of determining the best proximity points for these mappings. This foundational idea was further developed in 2009 by Thagafi and Shahzad [1], who extended the contraction condition and obtained new results for the concept introduced.

In [13], Karpagam and Agrawal generalized the concept of cyclic contractions, originally defined for two subsets to a framework involving $p \geq 2$ nonempty subsets by formulating the notion of the p-cyclic Meir-Keeler contraction based on the p-cyclic condition. In 2010, Vetro [20] contributed further to this theory by presenting new results in the context of p-cyclic ϕ-contractions. Moreover, various other generalizations and extensions of p-cyclic mappings have been discussed in works such as [10, 16] and several related studies [2, 5, 21].

Motivation and significance of the work. Best proximity point theory has been widely developed through cyclic contractions and their various generalizations. Nevertheless, most of the existing results are restricted to two subsets and a single mapping. Although cyclic contractions on p nonempty subsets have been studied, semi-cyclic contraction pair of mappings have been considered only in the case of two subsets. In this direction, Gabeleh and Abkar [9] pioneered the term called semi-cyclic contractions for a pair of operators (S, T). Later, Thakur et al. [19] further extended this framework by establishing best proximity point results for semi-cyclic ϕ-contraction pairs. Based on the existing results available to us, no previous work has considered semi-cyclic contraction pairs of mappings for $p \geq 2$ nonempty subsets. To overcome this gap, we introduce the notion of a p-semi-cyclic contraction pair of mappings and investigate best proximity point results for this new class of mappings. We establish corresponding existence and convergence theorems along with an illustrative example. The proposed framework significantly extends several existing contraction concepts. In particular, when $p = 2$, a p-semi-cyclic contraction pair reduces to a semi-cyclic contraction pair [9], and further, when $S = T$, it reduces to a cyclic contraction [6].

2. Preliminaries

We begin by presenting some fundamental definitions and results that will be essential for the development subsequent part of the paper.

“A Banach space X is said to be uniformly convex if there exists a strictly increasing function $\delta : [0, 2] \to [0, 1]$ such that the following implication holds for all $x, y, p \in X, R > 0$ and $r \in [0, 2R]$:

$$\|x - p\| \le R, \quad \|y - p\| \le R, \quad \|x - y\| \ge r \quad \Rightarrow$$
$$\Rightarrow \quad \left\| \frac{x+y}{2} - p \right\| \le \left(1 - \delta\left(\frac{r}{R}\right)\right) R.\text{“}$$

Definition 1 ([15])**.** Let (X, d) be a metric space, and let A and B be nonempty subsets of X. A mapping $T : A \cup B \to A \cup B$ is called a cyclic mapping if $T(A) \subseteq B$ and $T(B) \subseteq A$.

Definition 2 ([9])**.** Let A and B be nonempty closed subsets of complete metric space (X, d), and let S, T be two self-mapping on $A \cup B$. The pair (S, T) is called a semi-cyclic contraction pair if $S(A) \subseteq B$ and $T(B) \subseteq A$, and there exists $0 < \alpha < 1$ such that $d(Sx, Ty) \le d(x, y) + (1 - \alpha)d(A, B)$ for all $x \in A$ and $y \in B$.

Definition 3 ([6])**.** A subsets E of a metric space (X, d) is called boundedly compact if every bounded sequence in E has a subsequence that converges to a point in E.

3. Main results

Let us first introduce p-semi-cyclic map and p-semi-cyclic contraction pair.

Definition 4. Let A_i, $1 \le i \le p$ be nonempty closed subsets of a complete metric space (X, d) where p is positive even number. Let $S, T : \cup_{i=1}^{p} A_i \to \cup_{i=1}^{p} A_i$ be two self maps. Then the pair (S, T) is called a p-semi-cyclic pair if

(1) $$S(A_{2k-1}) \subseteq A_{2k} \quad \text{and} \quad T(A_{2k}) \subseteq A_{2k+1},$$

where $A_{p+1} = A_1$ and $k = 1, 2, \dots, \frac{p}{2}$.

Now, throughout the paper we consider A_i and mappings S and T defined as in Definition 4.

Definition 5. The pair (S, T) on $\cup_{i=1}^{p} A_i$ is called p-semi-cyclic contraction pair if there exists $\alpha \in (0, 1)$ such that

(2) $$d(Sx, Ty) \le \alpha d(x, y) + (1 - \alpha)d(A_i, A_{i+1}),$$

for all $x \in A_i$, $y \in A_{i+1}$.

Remark 1. (1) Note that, for $p = 2$, a p-semi-cyclic contraction pair coincides with the semi-cyclic contraction [9], and when $p = 2$ and $S = T$, the same is coincides with the cyclic contraction [6].

(2) According to the definition 4, we observe that the mappings S and T operates alternate across the subsets A_i, $1 \leq i \leq p$. That is, S applied on odd-indexed subsets, and T applied on even-indexed subsets (or vice versa), in a repeating fashion. For the sequence of mappings to complete a full cycle and return to the initial subset A_1 with the same mapping that started the process (say S), the total number of steps (i.e., the number of subsets p) must be even. Indeed, each application of S is followed by an application of T, and vice versa forming complete pairs. If p were odd, after p steps, the mapping that appears at A_1 would be different from the one that started the cycle. This breaks the alternating structure required by the p-semi-cyclic pairs. Therefore, to maintain the alternation and ensure that the cycle closes properly with the same structure (starting and ending with S or T), the number of subsets p must be even.

The following example illustrates that there exists a p-semi-cyclic contraction pair which is not cyclic.

Example 1. Let $X := \mathbb{R}^2$ and the norm given by $\|(x, y)\| = \max\{|x|, |y|\}$, where $(x, y) \in \mathbb{R}^2$. Let

$$A_1 = \Big\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq \tfrac{1}{2},\ y = 0\Big\},$$
$$A_2 = \Big\{(x, y) \in \mathbb{R}^2 : x = \frac{1}{2},\ \tfrac{1}{2} \leq y \leq 1\Big\},$$
$$A_3 = \Big\{(x, y) \in \mathbb{R}^2 : 1 \leq x \leq \tfrac{3}{2},\ y = 1\Big\},$$
$$A_4 = \Big\{(x, y) \in \mathbb{R}^2 : x = 1,\ 0 \leq y \leq \tfrac{1}{2}\Big\}.$$

Then each A_i is closed and $d(A_i, A_{i+1}) = \frac{1}{2}$, for all $1 \leq i \leq 4$. Define $S, T : \cup_{i=1}^4 A_i \to \cup_{i=1}^4 A_i$ by

$$S(x, y) = \begin{cases} (\frac{1}{2}, \frac{1}{2}), & (x, y) \in A_1; \\ (1, \frac{1}{2}), & (x, y) \in A_3; \\ (x, y), & \text{otherwise}; \end{cases}$$

and

$$T(x, y) = \begin{cases} (1, 1), & (x, y) \in A_2; \\ (\frac{1}{2}, 0), & (x, y) \in A_4; \\ (x, y), & \text{otherwise.} \end{cases}$$

Here, we observe that the pair (S, T) satisfies p-semi-cyclic map condition, however, neither T nor S is p-cyclic.

On the other hand, if $(x_i, x_{i+1}) \in A_i$ for all $1 \le i \le 4$. Define $S(x_1, y_1) = a_1$, $T(x_2, y_2) = a_2$, $S(x_3, y_3) = a_3$ and $T(x_4, y_4) = a_4$. Then $\|a_i - a_{i+1}\| = \frac{1}{2}$ where for all $1 \le i \le 4$ and $a_5 = a_1$.

Further, suppose $\|(x_i, y_i) - (x_{i+1}, y_{i+1})\| = N \ge 0$. Then

$$\|a_i - a_{i+1}\| = \frac{1}{2} \le \frac{1}{2}N + \frac{1}{2}d(A_i, A_{i+1}).$$

Hence, the pair (S, T) is a p-semi-cyclic contraction pair.

Next, we define the notion of p-semi-cyclic non-expansive mappings for pair (S, T).

Definition 6. The pair (S, T) on $\cup_{i=1}^{p} A_i$ is called p-semi-cyclic non-expansive mappings if

$$d(Sx, Ty) \le d(x, y) \text{ for all } x \in A_i,\ y \in A_{i+1} \text{ and } 1 \le i \le p. \tag{3}$$

The result below shows that the distance between adjacent sets are equal under p-semi-cyclic non-expansive mapping for (S, T).

Lemma 1. *Let the pair (S, T) is p-semi-cyclic nonexpansive mappings. Then $d(A_i, A_{i+1}) = d(A_{i+1}, A_{i+2})$ for all $1 \le i \le p$.*

Proof. Let $x \in A_i$ and $y \in A_{i+1}$, $1 \le i \le p$. Then, from (3), we have

$$d(A_{i+1}, A_{i+2}) \le d(Sx, Ty) \le d(x, y).$$

Since x and y are arbitrary elements of A_i and A_{i+1} respectively, it follows that

$$d(A_{i+1}, A_{i+2}) \le d(A_i, A_{i+1}), \quad \text{for all } i.$$

Applying this relation iteratively along the cycle, we obtain

$$d(A_2, A_3) \le d(A_3, A_4) \le \cdots d(A_p, A_1) \le d(A_1, A_2) \le d(A_2, A_3).$$

which implies that all the inequalities are actually equalities. Hence,

$$d(A_i, A_{i+1}) = d(A_1, A_2), \quad \text{for all } 1 \le i \le p.$$ □

Now we prove some essential results required for the subsequent development.

Proposition 1. *Let (S, T) be a p-semi-cyclic contraction pair. Consider $x_0 \in A_1$, and define*

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = Sx_n, \end{cases} \quad n = 0, 1, 2, \ldots \tag{4}$$

Then $\{x_n\}$ and $\{y_n\}$ are sequences in A_{2k-1} and A_{2k}, where $1 \le k \le \frac{p}{2}$, respectively. Moreover $d(x_n, Sx_n) \to d_0$ and $d(y_n, Ty_n) \to d_0$ as $n \to \infty$.

Proof. Consider $x_0 \in A_1$. Then $Sx_0 \in A_2$, so $\exists\, y_0 \in A_2$ such that $y_0 = Sx_0$. Now, $Ty_0 \in A_3$, so $\exists\, x_1 \in A_3$ such that $Ty_0 = x_1$. Further, $Sx_1 \in A_4$, so there exists $y_1 \in A_4$ such that $y_1 = Sx_1$. Analogously, we define the sequences $\{x_n\}$ and $\{y_n\}$ in A_{2k-1} and A_{2k}, where $1 \leq k \leq \frac{p}{2}$, respectively, as given in (4).

Moreover, from (2) we compute

$$\begin{aligned} d(x_{pn}, Sx_{pn}) &= d(Ty_{pn-1}, Sx_{pn}) \leq \alpha d(y_{pn-1}, x_{pn}) + (1-\alpha)d_0 \\ &= \alpha d(Sx_{pn-1}, Ty_{pn-1}) + (1-\alpha)d_0 \\ &\leq \alpha(\alpha d(x_{pn-1}, y_{pn-1}) + (1-\alpha)d_0) + (1-\alpha)d_0 \\ &= \alpha^2 d(x_{pn-1}, y_{pn-1}) + (1-\alpha^2)d_0 \\ &\vdots \\ &\leq \alpha^{2pn} d(x_0, y_0) + (1-\alpha^{2pn})d_0. \end{aligned}$$

Therefore, $d(x_{pn}, Sx_{pn}) \to d_0$.

Similarly, it can show that $d(y_{pn}, Ty_{pn}) \to d_0$. □

Proposition 2. *Let (S,T) be a p-semi-cyclic contraction. Assume, for $x_0 \in A_i$, sequences $\{x_n\}$ and $\{y_n\}$ generated by (4) have a convergent subsequence in A_{2k-1} and A_{2k} (where $1 \leq k \leq \frac{p}{2}$), respectively, then there exists $x \in A_{2k-1}$ and $y \in A_{2k}$ such that $d(x, Sx) = d_0 = d(y, Ty)$.*

Proof. Let sequence $\{x_{pn}\}$ have a convergent subsequence $\{x_{pn_k}\}$ which converge to x. Since

$$d_0 \leq d(Sx_{pn_k}, x) \leq d(Sx_{pn_k}, x_{pn_k}) + d(x_{pn_k}, x),$$

it follows from proposition 1 that $d(Sx_{pn_k}, x) \to d_0$. Also

$$\begin{aligned} d_0 \leq d(Sx, x_{pn_k+1}) &= d(Sx, Ty_{pn_k}) \leq \alpha d(x, y_{pn_k}) + (1-\alpha)d_0 \\ &= \alpha d(x, Sx_{pn_k}) + (1-\alpha)d_0. \end{aligned}$$

Letting $k \to \infty$, we have $d(Sx, x) = d_0$.

Similarly, we can prove that $d(Ty, y) = d_0$. □

Now, to illustrate proposition 1 and 2 we recall the example 1.

Example 2. Let X, A_1, A_2, A_3, A_4 are defined as in Example 1. Then the pair (S,T) p-semi-cyclic contraction. Let $x_0 \in A_1$ and apply $x_{n+1} = Ty_n$, $y_n = Sx_n$, then we have

$$\begin{aligned} &y_0 = S(x_0) = (\tfrac{1}{2}, \tfrac{1}{2}) \in A_2, && x_2 = T(y_1) = (\tfrac{1}{2}, 0) \in A_1, \\ &x_1 = T(y_0) = (1,1) \in A_3, && y_2 = S(x_2) = (\tfrac{1}{2}, \tfrac{1}{2}) \in A_2, \\ &y_1 = S(x_1) = (1, \tfrac{1}{2}) \in A_4, && x_3 = T(y_2) = (1,1) \in A_3, \text{ and so on.} \end{aligned}$$

Thus, the sequence satisfy $\{x_n\} \subset A_{2k-1}$ and $\{y_n\} \subset A_{2k}$, where $1 \leq k \leq 2$. Moreover, we observe that

$$d(x_n, Sx_n) = d(y_n, Ty_n) = \tfrac{1}{2} = d_0 \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Hence, the sequences converge to $d_0 = \frac{1}{2}$, as required by Proposition 1. Furthermore, by Proposition 2, there exist $x = (\frac{1}{2}, 0) \in A_1$, $y = (\frac{1}{2}, \frac{1}{2}) \in A_2$ such that $d(x, Sx) = d(y, Ty) = d_0 = \frac{1}{2}$.

Next result establishes the boundedness of sequences generated by a p-semi-cyclic contraction under the given conditions.

Proposition 3. *Let (S, T) be a p-semi-cyclic contraction. Assume, for $x_0 \in A_i$, sequences $\{x_n\}$ and $\{y_n\}$ generated by (4) are bounded.*

Proof. Since $d(x_{pn}, Sx_{pn}) \to d_0$, it is sufficient to prove the sequence $\{Sx_n\}$ is bounded in A_{2k}. If not, for each $M > 0$ $\exists N$ such that

$$M < d(x_p, Sx_{pN}) \quad \text{and} \quad d(x_p, Sx_{p(N-1)}) \leq M,$$

where

$$M > \max\left\{ \frac{d(x_0, Sx_0)}{\frac{1}{\alpha^{2p}} - 1} + \frac{(2p-2) + \frac{1}{\alpha^{2p}}}{\frac{1}{\alpha^{2p}} - 1} d_0, d(x_p, Sx_0) \right\}.$$

Moreover, we compute

$$\begin{aligned}
M < d(x_p, Sx_{pN}) &= d(Ty_{p-1}, Sx_{pN}) \\
&\leq \alpha d(y_{p-1}, x_{pN}) + (1-\alpha) d_0 \\
&= \alpha d(Sx_{p-1}, Ty_{pN-1}) + (1-\alpha) d_0 \\
&\leq \alpha(\alpha d(x_{p-1}, y_{pN-1}) + (1-\alpha) d_0) + (1-\alpha) d_0 \\
&= \alpha^2 d(x_{p-1}, y_{pN-1}) + (1-\alpha^2) d_0 \\
&\;\;\vdots \\
&= \alpha^{2p} d(x_0, y_{pN-p}) + (1-\alpha^{2p}) d_0,
\end{aligned}$$

thus, we obtain the inequality

$$M < \alpha^{2p} d(x_0, Sx_{pN-p}) + (1-\alpha^{2p}) d_0.$$

Therefore

$$\frac{1}{\alpha^{2p}}(M - (1-\alpha^{2p}) d_0) < d(x_0, Sx_{pN-p}),$$

consequently

$$\frac{1}{\alpha^{2p}}(M - d_0) + d_0 < d(x_0, x_p) + d(x_p, Sx_{pN-p}) \leq d(x_0, x_p) + M. \tag{5}$$

Now, from (2) and Proposition 1, we conclude that

$$\begin{aligned}
&d(x_0, x_p) \le \\
&\le d(x_0, Sx_0) + d(Sx_0, Ty_0) + d(Ty_0, Sx_1) + d(Sx_1, Ty_1) + \cdots \\
&\quad \cdots d(Ty_{p-2}, Sx_{p-1}) + d(Sx_{p-1}, Ty_{p-1}) \\
&\le d(x_0, Sx_0) + \big[\alpha d(x_0, y_0) + (1-\alpha)d_0\big] + \big[\alpha d(x_1, y_0) + (1-\alpha)d_0\big] \\
&\quad + \big[\alpha d(x_1, y_1) + (1-\alpha)d_0\big] + \cdots + \big[\alpha d(x_{p-1}, y_{p-1}) + (1-\alpha)d_0\big] \\
&\le d(x_0, Sx_0) + \big[\alpha d(x_0, Sx_0) + (1-\alpha)d_0\big] + \big[\alpha d(Ty_0, y_0) + (1-\alpha)d_0\big] \\
&\quad + \big[\alpha d(x_1, Sx_1) + (1-\alpha)d_0\big] + \cdots + \big[\alpha d(x_{p-1}, Sx_{p-1}) + (1-\alpha)d_0\big] \\
&\le d(x_0, Sx_0) + (2p-1)d_0.
\end{aligned}$$

Again, from inequality (5), we obtain

$$\frac{1}{\alpha^{2p}}(M - d_0) + d_0 < d(x_0, Sx_0) + (2p-1)d_0 + M,$$

imply that

$$M\Big(\frac{1}{\alpha^{2p}} - 1\Big) < d(x_0, Sx_0) + \Big((2p-2) + \frac{1}{\alpha^{2p}}\Big)d_0$$

or

$$M < \frac{d(x_0, Sx_0)}{\frac{1}{\alpha^{2p}} - 1} + \frac{(2p-2) + \frac{1}{\alpha^{2p}}}{\frac{1}{\alpha^{2p}} - 1}d_0,$$

a contradiction.

By similar arguments we can show that $\{y_n\}$ is bounded. □

Now we demonstrate an existence results under the assumption of boundedly compact sets.

Theorem 2. *Let (S, T) be a p-semi-cyclic contraction. Assume, for $x_0 \in A_i$, sequences $\{x_n\}$ and $\{y_n\}$ generated by (4) have a convergent subsequence in A_{2k-1} and A_{2k} (where $1 \le k \le \frac{p}{2}$), respectively. If atleast one A_i is boundedly compact then there exists $z \in \cup_{i=1}^{p} A_i$ such that either $d(z, Sz) = d_0$ or $d(z, Tz) = d_0$.*

Proof. Propositions 2 and 3 immediately lead to this outcome. □

Corollary 1. *Let (S, T) be a p-semi-cyclic contraction. Assume, for $x_0 \in A_i$, sequences $\{x_n\}$ and $\{y_n\}$ generated by (4) have a convergent subsequence in A_{2k-1} and A_{2k} (where $1 \le k \le \frac{p}{2}$), respectively. If span of at least one A_i is a finite dimensional subspace of X, then there exists $z \in \cup_{i=1}^{p} A_i$ such that either $d(z, Sz) = d_0$ or $d(z, Tz) = d_0$.*

Proof. As every finite dimensional subspace is boundedly compact, the result is a consequence of Theorem 2. □

We now proceed to the next result which shows the existence of the fixed point.

Theorem 3. *Let A_i, $1 \leq i \leq p$, be nonempty closed convex subset of uniformly convex Banach space X and let (S,T) be p-semi-cyclic contraction mapping. If $d_0 = 0$ then (S,T) has a unique common fixed point in $\cap_i^p A_i$.*

Proof. Suppose $d_0 = 0$. Then we have

$$\|Sx - Ty\| \leq \alpha\|x - y\| \quad \text{for all } x \in A_i,\ y \in A_{i+1}.$$

Consider sequences $\{x_n\}$ and $\{y_n\}$ as defined in (4). First we shall show that $\{x_{pn}\}$ and $\{y_{pn}\}$ are Cauchy sequence in $\{A_{2k-1}\}$ and $\{A_{2k}\}$, where $1 \leq k \leq \frac{p}{2}$, respectively. For, we compute

$$\begin{aligned}\|Sx_{pn+1} - Ty_{pn}\| &\leq \alpha\|x_{pn+1} - y_{pn}\| = \alpha\|Ty_{pn} - Sx_{pn}\| \\ &\leq \alpha^2\|y_{pn} - x_{pn}\| \leq \cdots \\ &\leq \alpha^{2pn}\|y_1 - x_1\| \to 0, \quad \text{as } n \to \infty.\end{aligned}$$

Then for $m > n$, we have

$$\|Sx_{pm} - Sx_{pn}\| \leq \sum_{k=n}^{m-1} \alpha^{2pn}\|y_1 - x_1\| \to 0, \quad n, m \to \infty.$$

Thus, $\{x_{pn}\}$ is a Cauchy sequence in $\{A_{2k-1}\}$, where $1 \leq k \leq \frac{p}{2}$.

In a similar manner, again for $m > n$, we have

$$\|Ty_{pm} - Ty_{pn}\| \leq \sum_{k=n}^{m-1} \alpha^{2pn}\|y_1 - x_1\| \to 0, \quad n, m \to \infty.$$

Thus, $\{y_{pn}\}$ is a Cauchy sequence in $\{A_{2k}\}$, where $1 \leq k \leq \frac{p}{2}$.

Since each A_i is closed in complete space X then there exists $x \in A_{2k-1}$ such that $x_{pn} \to x$ as $n \to \infty$. Since $x_{pn} \in A_{2k-1}$ then $Sx_{pn} \in A_{2k}$, so there exists $y_{pn} \in A_{2k}$ such that $Sx_{pn} = y_{pn}$ converges to Sx in A_{2k}. Further, $y_{pn} \in A_{2k}$, so there exits $x_{pn+1} \in A_{2k}$ such that $Ty_{pn} = x_{pn+1}$ converges to x in A_{2k}. Analogously, x converges to $\{A_{2k-1}\}$ and Sx converges to $\{A_{2k}\}$. It follows from Proposition 2 that, x is a common fixed point of S and T, i.e., $Sx = x = Tx$.

Next to show the uniqueness, let us assume that $y \neq x$ is such that $Ty = y = Sy$ for some $y \in \cap_i^p A_i$, then

$$\|x - y\| = \|Sx - Ty\| \leq \alpha\|Sx - Ty\|,$$

a contradiction, which completes the proof. □

Theorem 4. *Let A_i, $1 \leq i \leq p$ be nonempty closed convex subsets of a uniformly convex Banach space X and let pair (S,T) be a p-semi-cyclic contraction mapping. For $x_0 \in A_i$, the sequence $\{x_n\}$ and $\{y_n\}$ are generated by (4). Then, $\{x_{pn}\}$ and $\{y_{pn}\}$ are Cauchy sequences.*

Proof. For $d_0 = 0$, Theorem 3 immediately lead to this outcome. Consider $d_0 > 0$. On contrary, there exists $\epsilon_0 > 0$ such that for each $k \geq 1$, there exists $m_k > n_k \geq k$ so that

$$\|x_{pm_k} - x_{pn_k}\| \geq \epsilon_0. \tag{6}$$

Choose $\epsilon > 0$ such that $(1 - \delta(\frac{\epsilon_0}{d_0+\epsilon}))(d_0 + \epsilon) < d_0$. By Proposition 1, we have

$$\|x_{pn_k} - y_{pn_k}\| = \|x_{pn_k} - Sx_{pn_k}\| \to d_0,$$

there exists a positive integer N such that

$$\|x_{pn_k} - y_{pn_k}\| < d_0 + \epsilon \quad \text{and} \quad \|x_{pm_k} - y_{pn_k}\| < d_0 + \epsilon \tag{7}$$

for all $m_k > n_k \geq N$. It follows from the uniform convexity of X and (6)–(7) that

$$\|\frac{x_{pm_k} + x_{pn_k}}{2} - y_{pn_k}\| \leq \big(1 - \delta(\frac{\epsilon_0}{d_0 + \epsilon})\big)(d_0 + \epsilon).$$

for all $m_k > n_k \geq N$. Since A_i is convex $\frac{x_{pm_k}+x_{pn_k}}{2} \in A_i$, the choice of ϵ and the fact that δ is strictly increasing imply that $\|\frac{x_{pm_k}+x_{pn_k}}{2} - y_{pn_k}\| < d_0$, for all $m_k > n_k \geq N$, a contradiction. Thus $\{x_{pn}\}$ is a Cauchy sequence in A_{2k-1}. By a similar argument we can show that $\{y_{pn}\}$ is a Cauchy sequence in A_{2k}. □

Theorem 5. *Let A_i, $1 \leq i \leq p$, be nonempty closed convex subset of a uniformly convex Banach space X and let (S, T) be a p-semi-cyclic contraction. Assume, for $x_0 \in A_i$, sequences $\{x_n\}$ and $\{y_n\}$ generated by (4). Then, there exists unique $x \in A_{2k-1}$ and $y \in A_{2k}$ (where $1 < k < \frac{p}{2}$) such that $\{x_{pn}\} \to x$, $\{y_{pn}\} \to y$ and $\|x - Sx\| = d_0 = \|y - Ty\|$.*

Proof. By Theorem 4 $\{x_{pn}\}$ is a cauchy sequence in A_{2k} and since A_{2k} is closed there exists $x \in A_{2k}$ such that $\{x_{pn}\}$ converges to x. It follows from Proposition 2 that $\|x - Sx\| = d_0$. Similarly we can show that the sequence $\{y_{pn}\}$ is converges to y in A_{2k-1} and $\|y - Ty\| = d_0$, where $1 < k < \frac{p}{2}$. For the uniqueness, assume that there exists another $x' \in A_{2k}$, $1 < k < \frac{p}{2}$ such that $\|x' - Sx'\| = d_0$. Since

$$d_0 \leq \|TSx - Sx\| \leq \alpha\|Sx - x\| + (1 - \alpha)d_0 = d_0,$$

it follows that $\|TSx - Sx\| = d_0 = \|x - Sx\|$, this gives $TSx = x$. Similarly, we can prove $TSx' = x'$. Now, if $x' \neq x$, then $d_0 < \|x - Sx'\|$, we obtain

$$\begin{aligned}
\|Sx - x'\| &= \|Sx - TSx'\| \leq \alpha\|x - Sx'\| + (1 - \alpha)d_0 \\
&< \|x - Sx'\| + (1 - \alpha)\|x - Sx'\| \\
&= \|x - Sx'\| = \|TSx - Sx'\| \\
&\leq \alpha\|Sx - x'\| + (1 - \alpha)d_0 \leq \|Sx - x'\|,
\end{aligned}$$

a contradiction. Hence $x = x'$.

Similarly, we can prove the uniqueness of $y \in A_{2k-1}$. □

4. Conclusion

The concept of a p-semi-cyclic contraction for a pair of mappings (S, T) defined on the union of $p(\geq 2)$ nonempty subsets in a Banach space was presented in this study. Both the cyclic contractions taken into consideration by Eldred and Veeremani [6] and the semi-cyclic contractions examined by Gabeleh [9] are extended by this framework. We expanded the application of best proximity theory by establishing fundamental existence and convergence theorems for best proximity points connected to (S, T). Additionally, the examples given show that the suggested framework is valid.

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